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Light and Sound Emissions from Nonlinear Plasma Fluctuations

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Light and sound emissions due to nonlinear fluctuations of an electron-plasma field are investigated from the viewpoint of interaction phenomena. The method of solution is similar to that of a forced-oscillation problem, where the equivalent force term couples the different order of the field amplitudes, and is also responsible for the energy conversion from transverse field to longitudinal and inversely. For the general solution, an nth-order perturbation theory is initiated, based on the nonlinear equations of the plasma field, and a formal solution for the nth order is derived. Analysis is performed for the lowest order nonlinear fluctuations; radiation field intensities and scattering cross sections for all possible types of interactions are obtained. Furthermore, it is shown that the scattered waves radiate with combination frequencies.

I. INTRODUCTION

IN recent years a considerable amount of attention has been given to the radiation phenomena from plasma fields.[1–3] The radiation by a large amplitude oscillation from a cold plasma,[4] radio emission from nonuniform plasmas,[5] scattering radiation due to the propagation of electromagnetic waves in a plasma medium[6] have been investigated. Methods of approach to the study of the phenomena have been varied. For example, the problem of the propagation of electromagnetic waves in a plasma, where there existed a combination scattering of these waves by density variations, was treated by a technique which considered the variaions of a conductivity tensor operator.[6] The scattering of electromagnetic waves from randomly distributed free electrons was investigated by Gordon,[7] and the subject has been separately treated by a number of authors.[8–10]

In most of these works, it has been assumed that electromagnetic or acoustical type radiations arise when incident waves are propagated in a medium. Alternatively, a varying current may be injected into the medium to generate such radiations.[11] Once the free propagation of possible waves is completely understood, it is usually a stiaightforward process to construct a model wherein the source can be either an incident wave or an injected varying current. In the present work, however, an entirely new argument will be developed, in which it will be shown that light and sound types of emissions can result from the self-interactions of possible plasma waves in the plasma field which can support large density fluctuations. Consequently, these interacting waves play the role of an equivalent source term for the generation of radiating waves in higher orders.

The field under study is an unbounded, collisionless, isotropic electron plasma, described by a set of nonlinear hydrodynamic and Maxwell equations where no external electric and magnetic fields are assumed. Since it is not necessary to directly solve this set of nonlinear equations, a perturbation method of nth order is initiated, and a systematic solution for multipole acoustical and electromagnetic radiations (from plasma fluctuations) is developed. The most essential features of the general nonlinear problem will be clearly revealed by means of quite simple mathematics, involving only second-order perturbation calculations. First-order approximation of the field equations defines the linear theory in which longitudinal and transverse waves propagate independently. It is in these second-order terms that coupling of plasma waves occurs. Consequently, if longitudinal and transverse waves propagate simultaneously, in the second order they will interact. It can be assumed that coupling of various plasma waves is due to the density inhomogeneities, and the inhomogeneity plays the role of conversion mechanism of energy from a longitudinal wave field to a transverse wave field, and inversely. It is shown that in the second order, the partial differential equation for the electric field vector will become an inhomogeneous one, of which the inhomogeneous part consists of the fiıst-order variables. Generally, for an nth-order equation of the electric field vector, the source (inhomogeneous part) consists of $(n-1)$th and lesser order terms. The source term couples not only the different orders of the field amplitudes, but is also responsible for the coupling of longitudinal and transverse plasma wave fields. The basic problem is to determine the lowest order nonlinear amplitudes (second order) which can be expressed by quantities calculated from the linear (first-order) equa-

[1] Continuum theory of waves and radiation in a plasma and discussed by M. H. Cohen [see Phys. Rev. **123**, 711 (1961); **126**, 389 (1962)], and the problem of density fluctuations in plasma has been investigated by E. E. Salpeter [see Phys. Rev. **120**, 1528 (1960); **122**, 1663 (1961) and J. Geophys. Res. **68**, 1321–1333 (1963)].
[2] V. L. Ginzburg and V. V. Zheleznіakov, Soviet Astron. A. J. **2**, 653–668 (1958).
[3] *Radiation and Waves in Plasmas*, edited by M. Mitchner (Stanford University Press, Stanford, California, 1961).
[4] D. A. Tidman and G. H. Weiss, Phys. Fluids **4**, 866 (1961).
[5] D. A. Tidman and G. H. Weiss, Phys. Fluids **4**, 703 (1961).
[6] A. I. Akhiezer, J. G. Prokhda, and A. G. Sitenko, Soviet Phys.—JETP **6**, 576 (1957).
[7] W. E. Gordon, Proc. Inst. Radio. Engrs. **46**, 1824 (1958).
[8] J. P. Dougherty and D. T. Farley, Proc. Roy. Soc. (London) **A259**, 79 (1960).
[9] J. A. Fejer, Can. J. Phys. **38**, 1114 (1960).
[10] M. H. Cohen, J. Geophys. Res. **67**, 2729 (1962).
[11] R. Karplus, Phys. Fluids **3**, 800 (1960).

tions. Similarly, higher order field variables may be obtained by terms of lesser orders.[12]

Light- and sound-type energy radiation intensities will be defined by one general expression which will be proportional to the second power of the volume of interaction, fourth power of frequency, and inversely proportional to the equilibrium density and the fifth power of the characteristic radiation velocity. An additional term will enter into the intensity expression of the radiating longitudinal wave field.

Depending on the nature of scattered waves, the characteristic radiation velocity and frequency terms will be different. For instance, in sound-type radiation, the characteristic velocity is adiabatic sound velocity; in light-type radiation, it is the velocity of light. A combination of interacting wave frequencies with the electron plasma frequency will appear in light-type intensity expressions, whereas sound-type radiation intensity expressions will contain a combination of the two interacting wave frequencies as well as the combination frequencies of light type.

In Sec. II, we discuss the plasma equations and give a general solution for the nth-order equation of the electric field vector. Section III contains the derivation and a discussion of the generalized Poynting theorem. In Sec. IV, a description of the method used in this investigation is given, and the radiating field components are obtained. Radiation intensities and scattering cross sections are derived and analyzed in Sec. V. Finally, a summary of results is given in Sec. VI.

II. PLASMA EQUATIONS

The plasma equations are obtained by applying the combined sets of hydrodynamic and Maxwellian equations to a completely ionized polarizable electron fluid. Equations are assumed to be valid for a system of charged particles which have electrical neutrality in the mean and sufficiently high-particle density to justify the passage from a discrete set of particles to a fluid medium. The plasma field is assumed to vary slowly enough in space and time for Lorentz equations to be replaced by Maxwell equations. Furthermore, the ions are considered to be fixed in space, so that their only effect is to electrically neutralize the plasma. A fundamental property of the plasma described here is that it exhibits a "screening" property. By screening, we mean that the plasma has the property of cancelling any externally imposed field and reducing it to zero for a distance of the order of Debye length. If the usual averaging process is carried out to replace the Lorentz equations by Maxwell equations (that is to say that the averaging region has its mean dimensions larger than Debye length), the plasma may be represented as an electrically neutral polarizable fluid. Therefore, the plasma model to be investigated assumes electrical neutrality in the mean, hydrodynamic continuity of the medium, and validity of Maxwell's field equations. Electrical neutrality implies that we cannot consider distances smaller than Debye length, and we place limitations on the particle density, their velocities, and the rapidity with which the fields vary in time and space. We should have these assumptions in mind when the results are elaborated.

We are thus considering a one-component, collisionless, isotropic, electron fluid; the parameters associated with ions do not occur in the equations. The governing equations are:

$$(\rho^{(0)}+\rho)[(\partial\mathbf{u}/\partial t)+\mathbf{u}\cdot\nabla\mathbf{u}] +\nabla p+(e/m)(\rho^{(0)}+\rho)[\mathbf{E}+(1/c)\mathbf{u}\times\mathbf{H}]=0\,, \quad (1)$$

$$\nabla p=v^2\nabla\rho\,, \quad (2)$$

$$c\nabla\times\mathbf{E}=-\partial\mathbf{H}/\partial t\,, \quad (3)$$

$$c\nabla\times\mathbf{H}=-4\pi(e/m)(\rho^{(0)}+\rho)\mathbf{u}+(\partial\mathbf{E}/\partial t)\,, \quad (4)$$

$$\nabla\cdot\mathbf{E}=-4\pi(e/m)\rho\,, \quad (5)$$

where the space ($\mathbf{r}$) and time (t) variations of the electron fluid density, electron fluid pressure, electron fluid velocity, electric field, and magnetic field vectors are represented by ρ, p, $\mathbf{u}$, $\mathbf{E}$, and $\mathbf{H}$, respectively. The five variables are assumed to have large amplitudes in the oscillations of the electron plasma. No drift velocities are assumed. $\rho^{(0)}$ stands for the constant equilibrium density of electrons; v is the adiabatic sound velocity in plasma medium; c is the velocity of light; e and m are the charge and the mass of electron, respectively; $[(e/m)\rho]$ is the varying charge density. The above equations are self-consistent for the five field variables, and the equation of conservation of charge density which is missing in the equations above can be derived from Eqs. (4) and (5).

By manipulations in Eqs. (1) to (5), we obtain the following equation:

$$\nabla\times\nabla\times\mathbf{E}+\frac{1}{c^2}\frac{\partial^2\mathbf{E}}{\partial t^2}-\frac{v^2}{c^2}\nabla(\nabla\cdot\mathbf{E})+\frac{\omega_e^2}{c^2}\mathbf{E} =\frac{4\pi(e/m)}{c^2}\left[\mathbf{u}\frac{\partial\rho}{\partial t}-\frac{e}{m}\rho\mathbf{E}-\rho^{(0)}\left(\mathbf{u}\cdot\nabla\mathbf{u}+\frac{e}{mc}\mathbf{u}\times\mathbf{H}\right) -\rho\left(\mathbf{u}\cdot\nabla\mathbf{u}+\frac{e}{mc}\mathbf{u}\times\mathbf{H}\right)\right], \quad (6)$$

where $\omega_e=(4\pi\rho^{(0)})^{1/2}(e/m)$ is the plasma Langmuir frequency. It is to be noted that $\rho^{(0)}$ represents a uniform background of charge and is assumed immobile. The field variables can be represented to arbitrarily

[12] A similar method is used by D. Montgomery to study nonlinear, time-dependent plasma oscillations with Boltzman's equation. See D. Montgomery, Phys. Rev. **123**, 1077 (1961). See also, D. Montgomery and D. A. Tidman, Phys. Fluids **7**, 242 (1964).

high orders as[6]

$$\rho(\mathbf{r},t)=\sum_{n=1}^{\infty}\rho^{(n)}(\mathbf{r},t),\quad p(\mathbf{r},t)=\sum_{n=1}^{\infty}p^{(n)}(\mathbf{r},t),$$

$$\mathbf{u}(\mathbf{r},t)=\sum_{n=1}^{\infty}\mathbf{u}^{(n)}(\mathbf{r},t),\quad (7)$$

$$\mathbf{E}(\mathbf{r},t)=\sum_{n=1}^{\infty}\mathbf{E}^{(n)}(\mathbf{r},t),\quad \mathbf{H}(\mathbf{r},t)=\sum_{n=1}^{\infty}\mathbf{H}^{(n)}(\mathbf{r},t),$$

where it is no loss of generality to assume that the amplitudes of the variables will be decreasing as the orders of perturbation are increasing, i.e., $\rho^{(1)}>\rho^{(2)}>\rho^{(3)}\cdots$.

Using vector identities, for each order of perturbation we are able to write from Eq. (6)

$$\left[\nabla^2-\frac{1}{c^2}\left(\frac{\partial^2}{\partial t^2}+\omega_e^2\right)\right]\mathbf{E}^{(n)}-\left(1-\frac{v^2}{c^2}\right)\nabla(\nabla\cdot\mathbf{E}^{(n)})=\mathbf{S}^{(n)},\quad (8)$$

where the inhomogeneous part $\mathbf{S}^{(n)}$ is composed of terms up to order $(n-1)$ only, and reads

$$\mathbf{S}^{(n)}=\frac{4\pi(e/m)}{c^2}\left\{\sum_{j=1}^{n-1}\left(-\mathbf{u}^{(n-j)}\frac{\partial\rho^{(j)}}{\partial t}+\frac{e}{m}\rho^{(j)}\mathbf{E}^{(n-j)}\right)+\rho^{(0)}\sum_{j=1}^{n-1}\left(\mathbf{u}^{(j)}\cdot\nabla\mathbf{u}^{(n-j)}+\frac{e}{mc}\mathbf{u}^{(j)}\times\mathbf{H}^{(n-j)}\right)\right.$$
$$\left.+\sum_{j=1}^{n-2}\rho^{(j)}\sum_{s=1}^{n-j-1}\left(\mathbf{u}^{(s)}\cdot\nabla\mathbf{u}^{(n-j-s)}+\frac{e}{mc}\mathbf{u}^{(s)}\times\mathbf{H}^{(n-j-s)}\right)\right\},\quad (9)$$

for $n>1$,

$\mathbf{S}^{(n)}=0$, for $n=1$,

so that Eq. (8) is homogeneous for the first order. This shows that a fluctuating plasma generates the same variations as those produced in a linearly oscillating medium by a system of externally applied forces, represented by $\mathbf{S}^{(n)}$. This treatment of the radiation, as generated by the plasma fluid in the manner of a forced oscillation, is suitable since not only the mathematics involved become straightforward, but the interaction picture will be automatically accounted for in the equivalent applied force system $\mathbf{S}^{(n)}$.

Furthermore, relationships between plasma field variables are needed. Substituting Eqs. (7) into Eqs. (1) and (5), the following relationships between plasma variables will be in order:

$$\rho^{(n)}=-\frac{1}{4\pi(e/m)}\nabla\cdot\mathbf{E}_L^{(n)},\quad p^{(n)}=-\frac{v^2}{4\pi(e/m)}\nabla\cdot\mathbf{E}_L^{(n)},\quad \frac{\partial\mathbf{H}^{(n)}}{\partial t}=-c\nabla\times\mathbf{E}_T^{(n)}$$

$$\frac{\partial\mathbf{u}^{(n)}}{\partial t}=-\frac{1}{\rho^{(0)}}\nabla p^{(n)}-\frac{e}{m}\mathbf{E}^{(n)}-\sum_{j=1}^{n-1}\left(\mathbf{u}^{(n-j)}\cdot\nabla\mathbf{u}^{(j)}+\frac{1}{\rho^{(0)}}\rho^{(n-j)}\frac{\partial\mathbf{u}^{(j)}}{\partial t}+\frac{e}{mc}\mathbf{u}_T^{(n-j)}\times\mathbf{H}_T^{(j)}+\frac{(e/m)}{\rho^{(0)}}\rho^{(n-j)}\mathbf{E}^{(j)}\right)$$
$$-\frac{1}{\rho^{(0)}}\sum_{j=1}^{n-2}\rho^{(j)}\sum_{s=1}^{n-j-1}\left(\mathbf{u}^{(s)}\cdot\nabla\mathbf{u}^{(n-j-s)}+\frac{e}{mc}\mathbf{u}_T^{(s)}\times\mathbf{H}_T^{(n-j-s)}\right),\quad (10)$$

where subscripts L and T are used for longitudinal (irrotational) and transverse (divergenceless) parts of the vector field.

By taking the Fourier transform of Eq. (8) with respect to time, we obtain

$$(\nabla^2+k^{T2})\mathbf{E}_\omega^{(n)}-(1-k^{T2}/k^{L2})\nabla(\nabla\cdot\mathbf{E}_\omega^{(n)})=\mathbf{S}_\omega^{(n)},\quad (11)$$

where

$$k^T=(\omega/c)\left(1-\frac{\omega_e^2}{\omega^2}\right)^{1/2},\quad\text{and}\quad k^L=(\omega/v)\left(1-\frac{\omega_e^2}{\omega^2}\right)^{1/2}$$

are the propagation constants for transverse and longitudinal wave fields, respectively, where the subscript ω refers to the Fourier-transformed component of the field variables.

We shall define the tensor Green's functions associated with a region of the field wherein the equivalent source is assumed to be located. Consider, therefore, that $\mathbf{E}_\omega^{(n)}(\mathbf{r})$ is produced by $\mathbf{S}_\omega^{(n)}(\mathbf{r})$. We therefore introduce the dyadic $\mathbf{\Gamma}(\mathbf{r},\mathbf{r}')$ defined by[13]

$$(\nabla^2+k^{T2})\mathbf{\Gamma}(\mathbf{r},\mathbf{r}')-(1-k^{T2}/k^{L2})\nabla\nabla\cdot\mathbf{\Gamma}(\mathbf{r},\mathbf{r}')=\mathfrak{J}\delta(\mathbf{r}-\mathbf{r}'),\quad (12)$$

where $\mathfrak{J}$ is the unit dyadic, and $\delta(\mathbf{r}-\mathbf{r}')$ is defined by

$$\int_{V_0}\delta(\mathbf{r}-\mathbf{r}')dv=1,\quad \delta(\mathbf{r}-\mathbf{r}')=0\quad |\mathbf{r}-\mathbf{r}'|\neq 0,\quad (13)$$

in which integration is to be extended over a region enclosing the point $\mathbf{r}'$. The dyadic Green's function for

[13] Solutions to similar vector wave equations are given by P. M. Morse and H. Feshbach, *Methods of Theoretical Physics* (McGraw-Hill Book Company, Inc., New York, 1953), and by H. Levine and J. Schwinger, Com. Pure Appl. Math. 3, 355 (1950).

an unbounded medium, defined by Eq. (12), can be constructed by imposing the requirement that all of its components vanish at infinity. If the source function is known, the electric field vector is given by

$$\mathbf{E}_\omega^{(n)}(\mathbf{r}) = -\int_{V_0} \mathbf{\Gamma}(\mathbf{r},\mathbf{r}')\cdot\mathbf{S}_\omega^{(n)}(\mathbf{r}')dv', \quad (14)$$

where the integral is taken over the volume in which the source term is located. The dyadic Green's function for such an unbounded medium can be expressed by

$$\mathbf{\Gamma}(\mathbf{r},\mathbf{r}') = (k^L/k^{T2})\mathbf{\Gamma}_L(\mathbf{r},\mathbf{r},\omega) + \mathbf{\Gamma}_T(\mathbf{r},\mathbf{r}',\omega),$$

$$\mathbf{\Gamma}_L(\mathbf{r},\mathbf{r}') = \frac{1}{k^{L2}}\nabla\nabla'\frac{\exp\{ik^L|\mathbf{r}-\mathbf{r}'|\}}{4\pi|\mathbf{r}-\mathbf{r}'|}, \quad (15)$$

$$\mathbf{\Gamma}_T(\mathbf{r},\mathbf{r}') = \left(\mathfrak{J} - \frac{1}{k^{T2}}\nabla\nabla'\right)\frac{\exp\{ik^T|\mathbf{r}-\mathbf{r}'|\}}{4\pi|\mathbf{r}-\mathbf{r}'|},$$

where $\mathbf{\Gamma}_L(\mathbf{r},\mathbf{r}')$ and $\mathbf{\Gamma}_T(\mathbf{r},\mathbf{r}')$ stand for longitudinal and transverse wave fields; $\mathbf{r}$ and $\mathbf{r}'$ represent the vectors from the origin to the observation point and point of source, respectively. $\mathbf{r}'$ lies in the inside of the volume of interaction, whereas $\mathbf{r}$ extends to the outside of the volume of interaction.

The field vector $\mathbf{E}_\omega^{(n)}(\mathbf{r})$ of any arbitrary order can be evaluated by computing the lower orders first. We start with the lowest order ($n=1$), in which the vector wave equation (8) is homogeneous and its solution is known. The next and higher orders are solvable, since the inhomogeneous part of the vector wave equation will contain terms of lower orders. Time domain solutions will be obtained by simply taking the inverse Fourier transform of the field vector $\mathbf{E}_\omega^{(n)}(\mathbf{r})$.

III. GENERALIZED POYNTING THEOREM

In classical electromagnetic theory, the Poynting vector represents the amount of energy which crosses per unit area per second, whose normal is oriented in the direction of the Poynting vector. This definition is generally used for the transverse electromagnetic field. Possible ambiguities or arbitrariness in the interpretation of the theorem can be eliminated by applying the definition cautiously, such as applying averages over small but finite regions of space and time.

The classical Poynting theorem needs to be generalized for the present compressible plasma model, so that the total flow of plasma energy in radial direction through a closed surface will be properly defined. The Poynting vector for this case will include a longitudinal component added to its known transverse component. Similar to classical procedure, from Eqs. (4) and (5), we write:

$$\nabla\cdot\left(\frac{c}{4\pi}\mathbf{E}\times\mathbf{H}\right) = -\frac{1}{4\pi}\mathbf{E}\cdot\frac{\partial\mathbf{E}}{\partial t} - \frac{1}{4\pi}\mathbf{H}\cdot\frac{\partial\mathbf{H}}{\partial t} - \mathbf{E}\cdot\mathbf{J}_{\text{con}}, \quad (16)$$

where $\mathbf{J}_{\text{con}} = -(e/m)(\rho^{(0)}+\rho)\mathbf{u}$ represents the convection current due to the motions of electrons. In ordinary electromagnetic theory, the last term is interpreted as the one which expresses power expended by the convective flow of charges against the impressed field vector $\mathbf{E}$. If all material bodies in the field were absolutely rigid, no possible transformation of electromagnetic energy into the longitudinal wave energy would occur. Since the plasma medium is not assumed to be rigid but compressible, energy conversion from transverse waves, and inversely, will be effective. Consequently, a flow of electromagnetic energy across the boundary will be accompanied by the energy flow due to the longitudinal field.

To define this new term, we consult the governing Eqs. (1) to (5). After multiplying both sides of Eq. (1) by $\mathbf{u}$ and using necessary vector identities, the following expression is obtained:

$$\mathbf{E}\cdot\mathbf{J}_{\text{con}} = (\rho^{(0)}+\rho)\mathbf{u}\cdot\partial\mathbf{u}/\partial t + (\rho^{(0)}+\rho)\mathbf{u}\cdot[(\mathbf{u}\cdot\nabla)\mathbf{u}] - \nabla\cdot(p\mathbf{u}) + p\nabla\cdot\mathbf{u}. \quad (17)$$

It is to be noted that the velocity vector $\mathbf{u}$ contains a longitudinal component as well as a transverse one, so that the two waves are coupled.

It is due to the term $\mathbf{E}\cdot\mathbf{J}_{\text{con}}$ that the two directional energy conversions (from transverse waves to longitudinal waves, and inversely) can easily be explained. Combining Eq. (17) with (16), the following expression will be in order:

$$\nabla\cdot\left(\frac{c}{4\pi}\mathbf{E}\times\mathbf{H} + p\mathbf{u}\right) = -\frac{1}{4\pi}\mathbf{E}\cdot\frac{\partial\mathbf{E}}{\partial t} - \frac{1}{4\pi}\mathbf{H}\cdot\frac{\partial\mathbf{H}}{\partial t} + (\rho^{(0)}+\rho)\mathbf{u}\cdot\frac{\partial\mathbf{u}}{\partial t} + (\rho^{(0)}+\rho)\mathbf{u}\cdot[(\mathbf{u}\cdot\nabla)\mathbf{u}] + p\nabla\cdot\mathbf{u}, \quad (18)$$

where the flows of transverse and longitudinal energy appear under the divergence operation on the left side. The terms including velocity vector $\mathbf{u}$, on the right-hand side, are significant to the description of the plasma flow, such as turbulence. Indeed $(\rho^{(0)}+\rho)\mathbf{u}\cdot\partial\mathbf{u}/\partial t$ is comparable to Reynold stresses of the classical hydrodynamic theory, which make possible direct transfer of momentum components by velocity components. They are responsible for the momentum transport.

These terms can be interpreted in a similar manner here. Since density variable contains the static part $\rho^{(0)}$, $\rho^{(0)}\mathbf{u}\cdot\partial\mathbf{u}/\partial t$ will be the rate of change of lower order Reynold stresses, whereas the rate of change of higher order Reynold stresses will be due to $\rho\mathbf{u}\cdot\partial\mathbf{u}/\partial t$.

The generalized expressions of the Poynting vector for higher order perturbations are written as

$$\mathbf{P}^{(n)} = \sum_{s=1}^{n}\left(\frac{c}{4\pi}\mathbf{E}_T^{(s)}\times\mathbf{H}_T^{(n-s)*} + p^{(s)}\mathbf{u}_L^{(n-s)*}\right), \quad (19)$$

where the asterisk denotes the complex conjugate. The physical importance of this general form of the Poynting vector lies in the fact that aside from expressing the energy flow resulting from the same orders of the different components, the energy flow resulting from coupling between different orders of the field variables is also included. However, this most general expression will be used in rather a simplified form. Contributions from coupled terms between different orders will disappear when time averages are taken, since cross terms will contain periodic time multipliers.

IV. DESCRIPTION OF THE METHOD USED AND RADIATING FIELD COMPONENTS

We assume that fluctuating electron fluid occupies a limited part of a very large volume of the plasma field, of which the remainder oscillates with small amplitudes. Fluctuations of higher order amplitudes can be looked upon as resulting from the interactions of linearly oscillating plasma waves. We shall develop a systematic solution by keeping this model in mind.

The equations describing the higher order fluctuations will be constructed from the field variables of the linear oscillations. One can assume that an element of the plasma fluid, subjected to the equivalent force term, will suffer both compressional and rotational deformations to generate light and sound types of radiations, The linearly oscillating field would experience a pressure field, varying with small amplitude from a simple hydrostatic pressure field. Thus the variations of the pressure field would be proportional to the variations in density; the constant of proportionality will be the square of the adiabatic sound velocity v^2.

Consistent with the method of approach described above, we shall start with the solution of the homogeneous equation. Separating longitudinal and transverse wave fields, from Eq. (8), we write

$$\left[\nabla^2-\frac{1}{c^2}\left(\frac{\partial^2}{\partial t^2}+\omega_e^2\right)\right]\mathbf{E}_T^{(1)}=0\,, \qquad (20)$$

$$\left[\nabla^2-\frac{1}{v^2}\left(\frac{\partial^2}{\partial t^2}+\omega_e^2\right)\right]\mathbf{E}_L^{(1)}=0\,, \qquad (21)$$

which are homogeneous Klein-Gordon type differential equations. The plane waves which may exist in the plasma are determined by assuming solutions to the above equations of the form $\mathbf{v}\cos(\omega t-\boldsymbol{\kappa}\cdot\mathbf{r})$. The propagation vector $\boldsymbol{\kappa}$ is $\hat{\kappa}(\omega/v)[1-(\omega_e^2/\omega^2)]^{1/2}$ for longitudinal waves and $\hat{\kappa}(\omega/c)[1-(\omega_e^2/\omega^2)]^{1/2}$ for transverse waves; $\hat{\kappa}$ is a unit vector in the direction of propagation and $\mathbf{r}$ is the vector to the point of observation. $\mathbf{v}=\nu_0\hat{\nu}$ is either parallel or perpendicular to the propagation vector $\boldsymbol{\kappa}$; that is, the primary wave is either longitudinal or transverse.

In these first-order perturbation solutions, one solution corresponds to a longitudinal wave which has no magnetic field associated with it the remaining solutions correspond to two transverse waves of perpendicular polarization which have no density variations associated with them.

We now turn our attention to the problem of finding the second-order field components. To analyze the subject clearly and relate it to emissions of sound and light as they are produced by an externally applied force field, we define the term $\mathbf{S}^{(2)}(\mathbf{r},t)$ as the source term per-unit-volume, which will be simply written as:

$$\mathbf{S}^{(2)}(\mathbf{r},t)=\frac{4\pi e/m}{c^2}\left(\mathbf{u}^{(1)}\frac{\partial\rho^{(1)}}{\partial t}+\frac{e}{m}\mathbf{E}^{(1)}+\rho^{(0)}\mathbf{u}^{(1)}\cdot\nabla\mathbf{u}^{(1)}+\frac{e\rho^{(0)}}{mc}\mathbf{u}^{(1)}\times\mathbf{H}^{(1)}\right). \qquad (22)$$

As mentioned before, this vector source expression couples transverse and longitudinal wave fields.

The medium is assumed to be unbounded, and at points far enough from the location of the interaction. The radiation field variables will be computed by using the far-field approximation. By this approximation, the dyadic Green's functions simplify into the following forms:

$$\boldsymbol{\Gamma}_L(\mathbf{r},\mathbf{r}',\omega)=\mathbf{n}_r\mathbf{n}_r\exp\{i(r-\mathbf{n}_r\cdot\mathbf{r}')k^L\}/4\pi r\,, \qquad (23)$$

$$\boldsymbol{\Gamma}_T(\mathbf{r},\mathbf{r}',\omega)=(\mathfrak{J}-\mathbf{n}_r\mathbf{n}_r)\exp\{i(r-\mathbf{n}_r\cdot\mathbf{r}')k^T\}/4\pi r\,, \qquad (24)$$

where $\mathbf{r}$ is chosen so large that $k^L|\mathbf{r}-\mathbf{r}'|\gg1$, $k^T|\mathbf{r}-\mathbf{r}'|\gg1$ for all $\mathbf{r}$; and in addition, since $|\mathbf{r}'|\ll|\mathbf{r}|$ we have $|\mathbf{r}-\mathbf{r}'|\simeq r-\mathbf{n}_r\cdot\mathbf{r}'$, where $\mathbf{n}_r$ is the unit vector in the direction of $\mathbf{r}$, i.e., $\mathbf{n}_r=\mathbf{r}/r$.

The lowest order source term $\mathbf{S}^{(2)}(\mathbf{r},t)$ will be computed by assuming that two primary plane waves (which are solutions of the homogeneous equation) with different frequencies and amplitudes are interacting. Therefore, we write

$$\mathbf{E}^{(1)}(\mathbf{r},t)=\sum_{j=1}^{2}\mathbf{v}_j\cos(\omega_j t-\boldsymbol{\kappa}_j\cdot\mathbf{r}), \qquad (25)$$

which also defines the other four field variables in the first-order perturbation, by the help of relationships given in Eq. (10).

Substituting the first-order field variables into Eq. (9), and after some lengthy calculations, the following expression is obtained:

$$\mathbf{S}^{(2)}(\mathbf{r},t)=-\frac{(e/m)}{c^2}\sum_{s\mp n}\mathbf{S}_{s\mp n}\sin[(\omega_s\mp\omega_n)t-(\boldsymbol{\kappa}_s\mp\boldsymbol{\kappa}_n)\cdot\mathbf{r}], \qquad (26)$$

where a summation rule is adopted. The subscript $(s\mp n)$ takes the value of $(1+1)$, $(2+2)$, $(1-2)$, and $(1+2)$, where $(1+1)$ and $(2+2)$ refer to the self-interactions of the same primary waves; whereas, $(1-2)$ and $(1+2)$ refer to interactions between the two different primary waves. Hereafter, these four different modes of $\mathbf{S}_{s\mp n}$ will be called interaction modes, and it will be understood that when $s\mp n=1+1$, $\omega_s\mp\omega_n$ reads

$\omega_1+\omega_1$; $\kappa_s\mp\kappa_n$ reads $\kappa_1+\kappa_1$; and thus $S_{s\mp n}$ becomes S_{1+1}. The four components of $S_{s\mp n}$ are given by

$$\mathbf{S}_{s\mp n}=\frac{1}{8\pi}\left\{\eta_{1s}(\boldsymbol{\eta}_{2n}+\mathbf{v}_n)+\frac{\omega_e^2}{\omega_s\omega_n}[(\boldsymbol{\eta}_{2s}\cdot\boldsymbol{\kappa}_n)\boldsymbol{\eta}_{2n}+(\boldsymbol{\eta}_{2s}\times\boldsymbol{\xi}_{2n})]\right\}\mp(1-\delta_s^n)$$
$$\times\frac{1}{8\pi}\left\{\eta_{1n}(\boldsymbol{\eta}_{2s}+\mathbf{v}_s)+\frac{\omega_e^2}{\omega_s\omega_n}[(\boldsymbol{\eta}_{2n}\cdot\boldsymbol{\kappa}_s)\boldsymbol{\eta}_{2s}+(\boldsymbol{\eta}_{2n}\times\boldsymbol{\xi}_{2s})]\right\},\quad(27)$$

where

$$\begin{aligned}&\eta_{1s}=\mathbf{v}_s\cdot\boldsymbol{\kappa}_s, && \eta_{1n}=\mathbf{v}_n\cdot\boldsymbol{\kappa}_n\\ &\boldsymbol{\eta}_{2s}=(v^2/\omega_e^2)(\mathbf{v}_s\cdot\boldsymbol{\kappa}_s)\boldsymbol{\kappa}_s+\mathbf{v}, && \boldsymbol{\eta}_{2n}=(v^2/\omega_e^2)(\mathbf{v}_n\cdot\boldsymbol{\kappa}_n)\boldsymbol{\kappa}_n+\mathbf{v}_n\\ &\boldsymbol{\xi}_{2s}=\boldsymbol{\kappa}_s\times\mathbf{v}_s, && \boldsymbol{\xi}_{2n}=\boldsymbol{\kappa}_n\times\mathbf{v}_n.\end{aligned}\quad(28)$$

Here, s and n indices are used separately with numbers 1 and 2 preceding them. The summation rule is valid again, and δ_s^n is 1 when $s=n$, and otherwise zero. It is furthermore worthwhile to remember that throughout this analysis, $\omega_s\,\omega_n\gg\omega_e$ is assumed.

The source term $\mathbf{S}^{(2)}(\mathbf{r},t)$ is determined by the parameters of the primary waves in Eq. (26). By using, now, Eqs. (23), (24), and (14) the following expressions of longitudinal and transverse field components can be written:

$$\mathbf{E}_L^{(2)}(\mathbf{r},t)=\sum_{s\mp n}\mathbf{E}_{L_{s\mp n}}^{(2)}(\mathbf{r},t)$$
$$=\frac{1}{2i}\frac{e/m}{v^2}\sum_{s\mp n}\boldsymbol{\alpha}_{s\mp n}^L\left(\frac{\exp\{-i[k_{s\mp n}^Lr-(\omega_s\mp\omega_n)t]\}}{r}a_{s\mp n}^L-\frac{\exp\{i[k_{s\mp n}^Lr-(\omega_s\mp\omega_n)t]\}}{r}a_{s\mp n}^{L*}\right),\quad(29)$$

$$\mathbf{E}_T^{(2)}(\mathbf{r},t)=\sum_{s\mp n}\mathbf{E}_{T_{s\mp n}}(\mathbf{r},t)$$
$$=\frac{1}{2i}\frac{e/m}{c^2}\sum_{s\mp n}\boldsymbol{\alpha}_{s\mp n}^T\left(\frac{\exp\{-i[k_{s\mp n}^Tr-(\omega_s\mp\omega_n)t]\}}{r}a_{s\mp n}^T-\frac{\exp\{i[k_{s\mp n}^Tr-(\omega_s\mp\omega_n)t]\}}{r}a_{s\mp n}^{T*}\right),\quad(30)$$

where the summation rule is used again. When $s\mp n=1-2$, $k_{s\mp n}^L$ reads k_{1-2}^L; $\boldsymbol{\alpha}_{s\mp n}^L$ reads $\boldsymbol{\alpha}_{1-2}^L$, etc., $\cdots$. Summations are therefore understood to be taken for four different interaction modes. New parameters, which appear in Eqs. (29) and (30), are given by the following:

$$\begin{aligned}&a_{s\mp n}^{L,T}=\int_{V_0}\exp(i\boldsymbol{\chi}_{s\mp n}^{L,T}\cdot\mathbf{r}')dv' && \boldsymbol{\alpha}_{s\mp n}^L=\mathbf{n}_r\mathbf{n}_r\cdot\mathbf{S}_{s\mp n},\\ &\boldsymbol{\chi}_{s\mp n}^{L,T}=k_{s\mp n}^{L,T}\mathbf{n}_r-(\boldsymbol{\kappa}_s\mp\boldsymbol{\kappa}_n), && \boldsymbol{\alpha}_{s\mp n}^T=(\mathfrak{J}-\mathbf{n}_r\mathbf{n}_r)\cdot\mathbf{S}_{s\mp n},\\ &k_{s\mp n}^L=\frac{\omega_s\mp\omega_n}{v}\epsilon_{s\mp n}, && \epsilon_{s\mp n}=\left[1-\frac{\omega_e^2}{(\omega_s\mp\omega_n)^2}\right]^{1/2},\\ &k_{s\mp n}^T=\frac{\omega_s\mp\omega_n}{c}\epsilon_{s\mp n}.\end{aligned}\quad(31)$$

By using the relationships between the various plasma field variables, pressure, longitudinal component of the velocity vector, and magnetic field component, the following expressions are obtained:

$$p^{(2)}(\mathbf{r},t)=\sum_{s\mp n}p_{s\mp n}^{(2)}(\mathbf{r},t)$$
$$=\frac{i}{8\pi}\sum_{s\mp n}\mathbf{n}_r\cdot\boldsymbol{\alpha}_{s\mp n}^L\frac{\partial}{\partial r}\left(\frac{\exp\{-i[k_{s\mp n}^Lr-(\omega_s\mp\omega_n)t]\}}{r}a_{s\pm n}^L-\frac{\exp\{i[k_{s\mp n}^Lr-(\omega_s\mp\omega_n)t]\}}{r}a_{s\mp n}^{L*}\right),\quad(32)$$

$$\mathbf{u}_L^{(2)}(\mathbf{r},t)=\sum_{s\mp n}\mathbf{u}_{L_{s\mp n}}^{(2)}(\mathbf{r},t)=-\frac{i}{8\pi\rho^{(0)}}\sum_{s\mp n}\frac{\boldsymbol{\alpha}_{s\mp n}^L}{(\omega_s\mp\omega_n)}\left(\frac{\partial^2}{\partial r^2}-\frac{\omega_e^2}{v^2}\right)$$
$$\times\left(\frac{\exp\{-i[k_{s\mp n}^Lr-(\omega_s\mp\omega_n)t]\}}{r}a_{s\mp n}^L-\frac{\exp\{i[k_{s\mp n}^Lr-(\omega_s\mp\omega_n)t]\}}{r}a_{s\mp n}^{L*}\right)$$
$$+\frac{4\pi}{\omega_s\mp\omega_n}\mathbf{T}_{s\mp n}(\exp\{[(\boldsymbol{\kappa}_s\mp\boldsymbol{\kappa}_n)\mathbf{r}-(\omega_s\mp\omega_n)t]\}+\exp\{i[(\boldsymbol{\kappa}_s\mp\boldsymbol{\kappa}_n)\mathbf{r}-(\omega_s\mp\omega_n)t]\}),\quad(33)$$

$$\mathbf{H}_T^{(2)}(\mathbf{r},t)=\sum_{s\mp n}\mathbf{H}_{T_{s\mp n}}^{(2)}(\mathbf{r},t)$$

$$=\frac{e/m}{2c}\sum_{s\mp n}\frac{\mathbf{n}_r\times\boldsymbol{\alpha}_{s\mp n}^T}{\omega_s\mp\omega_n}\frac{\partial}{\partial r}\left(\frac{\exp\{-i[k_{s\mp n}^Tr-(\omega_s\mp\omega_n)t]\}}{r}a_{s\mp n}^T+\frac{\exp\{i[k_{s\mp n}^Tr-(\omega_s\mp\omega_n)t]\}}{r}a_{s\mp n}^{T*}\right), \quad (34)$$

where

$$\mathbf{T}_{s\mp n}=[\eta_{1s}\boldsymbol{\eta}_{1n}\mp(1-\delta_s^{\,n})\eta_{1n}\boldsymbol{\eta}_{1s}]-\mathbf{S}_{L_{s\mp n}}^{(2)}. \quad (35)$$

Once the field components are defined, the Poynting vector will be determined according to the rules previously adopted.

$\mathbf{P}^{(2)}$ is the Poynting vector due to the primary waves. The next nonvanishing component of the Poynting vector will be $\mathbf{P}^{(4)}$, since $\mathbf{P}^{(3)}$ vanishes when time averages are taken. Indicating time averages by $\langle\ \rangle_t$, due to the lowest order nonlinear terms, the Poynting vector reads:

$$\mathbf{P}^{(4)}=\mathbf{P}_T^{(4)}+\mathbf{P}_L^{(4)}=\mathrm{Re}\langle(c/8\pi)\mathbf{E}_T^{(2)}\times\mathbf{H}_T^{(2)*}\rangle_t+\mathrm{Re}\langle\tfrac{1}{2}p^{(2)}\mathbf{u}_L^{(2)*}\rangle_t, \quad (36)$$

where the first and second terms on the right-hand side represent the transverse and longitudinal parts respectively.

By using Eqs. (30) to (34), the two components of the Poynting vector are obtained:

$$\mathbf{P}_{T_{s\mp n}}^{(4)}=\mathrm{Re}\langle(c/8\pi)\mathbf{E}_{T_{s\mp n}}^{(2)}\times\mathbf{H}_{T_{s\mp n}}^{(2)*}\rangle_t=\left[\frac{(e/m)^2}{16\pi c^3r^2}\epsilon_{s\mp n}\boldsymbol{\alpha}_{s\mp n}^T\times(\mathbf{n}_r\times\boldsymbol{\alpha}_{s\mp n}^T)\right]a_{s\mp n}^Ta_{s\mp n}^{T*}, \quad (37)$$

$$\mathbf{P}_{L_{s\mp n}}^{(4)}=\mathrm{Re}\langle\tfrac{1}{2}p_{s\mp n}^{(2)}\mathbf{u}_{L_{s\mp n}}^{(2)}\rangle=\left[\frac{(e/m)^2}{16\pi v^3r^2}\left(\frac{\omega_s\mp\omega_n}{\omega_e}\right)^2\epsilon_{s\mp n}\boldsymbol{\alpha}_{s\mp n}^L\right]a_{s\mp n}^La_{s\mp n}^{L*}+\left[\frac{1}{16\pi r^2}\frac{\mathbf{n}_r\cdot\boldsymbol{\alpha}_{s\mp n}^L}{(\omega_s\mp\omega_n)}\mathbf{T}_s{}^{\pm}{}_n\right]$$

$$\times\{[(k_{s\mp n}r)\,{}^{(r)}a_{s\mp n}^L+{}^{(i)}a_{s\mp n}^L]\cos(\chi_{s\mp n}^L\cdot\mathbf{r})-[{}^{(r)}a_{s\mp n}^L-{}^{(i)}a_{s\mp n}^L(k_{s\mp n}r)]\sin(\chi_{s\mp n}^L\cdot\mathbf{r})\}. \quad (38)$$

The left superscripts (r) and (i) refer to the real and imaginary parts of the integral variables attached. Scattering radiations for longitudinal and transverse wave fields will be realized if χ^L and χ^T expressions become identical to zero. That is to say, if such an $\mathbf{n}_r$ direction is defined, $a_{s\mp n}^T$, $a_{s\mp n}^{T*}$, $a_{s\mp n}^L$, $a_{s\mp n}^{L*}$, and ${}^{(r)}a_{s\mp n}^L$ will each become proportional to the volume of interaction (V_0); whereas ${}^{(i)}a_{s\mp n}^L$ and $\sin(\chi_{s\mp n}^L\cdot\mathbf{r})$ terms will vanish from the expressions given above. For other values of $\mathbf{n}_r$, the amplitudes of waves will be oscillating; for these directions, waves do not scatter but behave as diffracted waves.

As will be seen later, the directional condition for scattering described above is a necessary, but not a sufficient condition for scattering radiation. It only informs us if such a scattering occurs in a definite direction. Relative position of the propagation vectors of the interacting primary waves, or in the case of polarized primary waves, the relative position of polarization planes are other important deciding factors on the occurrence of such scatterings.

V. INTENSITY ANALYSIS OF LIGHT AND SOUND FIELDS AND SCATTERING CROSS SECTIONS

The significant radiation quantities which, under certain conditions, can be estimated by human eye and ear, are the intensities of light and sound type radiations at any point of the field and their frequency spectrums. Being more descriptive, we named acoustical and optical (or electromagnetic) type radiations as sound- and light-type radiations which are to be used more cautiously otherwise. As it can be seen later, the intensity expressions of sound and light emissions at any point will contain either adiabatic sound velocity or the velocity of light (or a combination of both velocities) to the fifth power in their denominators. We shall call pure sound and pure light those radiations where the intensity expressions contain only the fifth power of the adiabatic sound velocity and light velocity, respectively. Furthermore, these velocities are taken as the characteristic velocities of the associated radiations. Intermediate radiations between the pure radiations will contain the two velocities, such as third power of the light velocity and the second power of sound velocity. Visibility and audibility of these radiations are problems of a different type, and will not be discussed here.

Before we formulate the radiation intensities, sound and light radiation energy densities will be derived from the Poynting vector expressions. This will be done by taking the components of the time-averaged Poynting vectors in the $\mathbf{n}_r$ direction, and multiplying them by $r^2d\Omega$. Thus, the time averaged energy densities per solid angle $d\Omega$ are written (for each interaction mode):

$$dP_{Ts\mp n}^{(4)}=\mathbf{n}_r\cdot\mathbf{P}_{Ts\mp n}^{(4)}r^2d\Omega, \quad (39)$$

$$dP_{Ls\mp n}^{(4)}=\mathbf{n}_r\cdot P_{Ls\mp n}^{(4)}r^2d\Omega. \quad (40)$$

To obtain the total sound and light power outputs, one must integrate these expressions of over a sphere.

Intensities of energy flux per unit solid angle at a point of the plasma field are obtained by dividing both sides of Eqs. (39) and (40) by $d\Omega$:

$$I_{T_{s\mp n}}=\frac{V_0^2}{16\pi}\frac{(e/m)^2}{c^3}\epsilon_{s\mp n}\mathbf{n}_r\cdot[\boldsymbol{\alpha}_{s\mp n}{}^T\times(\mathbf{n}_r\times\boldsymbol{\alpha}_{s\mp n}{}^T)],\quad(41)$$

$$I_{L_{s\mp n}}=\frac{V_0^2}{16\pi}\frac{(e/m)^2}{v^3}\frac{(\omega_s\mp\omega_n)^2}{\omega_e^2}\epsilon_{s\mp n}(\mathbf{n}_r\cdot\boldsymbol{\alpha}_{s\mp n}{}^L)^2+\frac{V_0}{16\pi\rho^{(0)}}\left(\frac{r}{v}\right)\epsilon_{s\mp n}(\mathbf{n}_r\cdot\boldsymbol{\alpha}_{s\mp n}{}^L)(\mathbf{n}_r\cdot T_{s\mp n}),\quad(42)$$

where scattering resonance conditions $\chi_{s\mp n}{}^{L,T}=0$ are assumed to be satisfied.

Unlike the intensity expression of transverse field, the intensity of the longitudinal field contains an additional term, whose appearance is due to the existence of first-order terms in the velocity vector $\mathbf{u}^{(2)}(\mathbf{r},t)$. When the pressure terms $p^{(2)}(\mathbf{r},t)$ are multiplied by the velocity vector $\mathbf{u}^{(2)}(\mathbf{r},t)$, cross multiplication between first- and second-order terms yields this additional term. In our subsequent discussions, it will be seen that these radiations are of quadrupole type, within the limits of second-order perturbations.

The scattered intensity expressions refer only to the energy which actually escapes from the fluctuations as light and sound. However, depending on the type of the interacting waves, there will be differences in the intensities of emitted light and sound. Here, we have three different possible combinations of interacting waves, namely, longitudinal-longitudinal, transverse-transverse, and longitudinal-transverse types.

Case I. Two Longitudinal Primary Waves

For this case, the general expressions for the longitudinal and transverse components of intensity undergo certain simplifications. Since the fluctuating force term contains only waves of a longitudinal type, cross product terms vanish in source terms, and the intensity expressions take the following forms:

$$I_{T_{s\mp n}}{}^{(LL)}=\frac{V_0^2\omega_e^2(\omega_s^2-\omega_e^2)}{(8\pi)^4\rho^{(0)}c^5}\left(\frac{v}{c}\right)^{-2}(1-\epsilon_s^2)^{-1}\epsilon_{s\mp n}\nu_{0s}^2\nu_{0n}^2|\mathbf{S}_{s\mp n}{}^{(LL)}|^2\sin^2\theta_{s\mp n}{}^{(LL)},\quad(43)$$

$$I_{L_{s\mp n}}{}^{(LL)}=\frac{V_0^2(\omega_s^2-\omega_e^2)(\omega_s\mp\omega_n)^2}{2(8\pi)^3\rho^{(0)}v^5}(1-\epsilon_s^2)^{-1}(1-\epsilon_n^2)^{-1}\epsilon_{s\mp n}\nu_{0s}^2\nu_{0n}^2|\mathbf{S}_{s\mp n}{}^{(LL)}|^2\cos^2\theta_{s\mp n}{}^{(LL)}$$
$$+\frac{V_0(\omega_s^2-\omega_e^2)r}{2(8\pi)^3\rho^{(0)}v^3}(1-\epsilon_s^2)^{-1}(1-\epsilon_n^2)^{-1}\epsilon_{s\mp n}\nu_{0s}^2\nu_{0n}^2|\mathbf{S}_{s\mp n}{}^{(LL)}||\mathbf{T}_{s}{}^{\pm}{}_{n}{}^{(LL)}|\cos\theta_{s\mp n}{}^{(LL)}\cos\phi_{s\mp n}{}^{(LL)},\quad(44)$$

where

$$\mathbf{S}_{s\mp n}{}^{(LL)}=\left\{\left[\left(1+\frac{\omega_n^2}{\omega_e^2}\right)\frac{\omega_e^2}{\omega_s\omega_n}+\left(\frac{\omega_n^2-\omega_e^2}{\omega_s^2-\omega_e^2}\right)^{1/2}(\hat{\kappa}_s\cdot\hat{\kappa}_n)\hat{\kappa}_n\mp(1-\delta_s{}^n)\left[\left(1+\frac{\omega_s^2}{\omega_e^2}\right)\frac{\omega_e^2}{\omega_s\omega_n}\left(\frac{\omega_n^2-\omega_e^2}{\omega_s^2-\omega_e^2}\right)^{1/2}+(\hat{\kappa}_s\cdot\hat{\kappa}_n)\right]\hat{\kappa}_s\right\},\quad(45)$$

$$\mathbf{T}_{s\mp n}{}^{(LL)}=\left\{\left[\left(\frac{\omega_n}{\omega_s}-\frac{\omega_e^2}{\omega_n\omega_s}\right)-\left(\frac{\omega_n^2-\omega_e^2}{\omega_s^2-\omega_e^2}\right)^{1/2}(\hat{\kappa}_s\cdot\hat{\kappa}_n)\right]\hat{\kappa}_n\mp(1-\delta_s{}^n)\left[\left(\frac{\omega_s}{\omega_n}-\frac{\omega_e^2}{\omega_s\omega_n}\right)\left(\frac{\omega_n^2-\omega_e^2}{\omega_s^2-\omega_e^2}\right)^{1/2}-(\hat{\kappa}_s\cdot\hat{\kappa}_n)\right]\hat{\kappa}_s\right\}.\quad(46)$$

Superscripts refer to the primary waves whereas the subscript defines the type of the radiating wave, and θ^{LL} and ϕ^{LL} are the angles between the scattering direction and $\mathbf{S}_{s\mp n}{}^{(LL)}$, $\mathbf{T}_{s\mp n}{}^{(LL)}$ source vectors, respectively. It is to be noted that $\mathbf{S}_{s\mp n}{}^{(LL)}$ and $\mathbf{T}_{s\mp n}{}^{(LL)}$ vectors lie in the plane of propagation vectors $(\boldsymbol{\kappa}_1,\boldsymbol{\kappa}_2)$.

As a basis for the construction of the type of radiation expressed above, we assumed that the resonance scattering conditions are satisfied. These latter conditions deserve further comment. Squaring the terms in the resonance conditions $\chi_{s\mp n}{}^L=0$ and $\chi_{s\mp n}{}^T=0$:

$$|k_{s\mp n}{}^L|^2-|\boldsymbol{\kappa}_s|^2-|\boldsymbol{\kappa}_n|^2\pm2|\boldsymbol{\kappa}_s||\boldsymbol{\kappa}_n|\cos\psi_{L_{s\mp n}}{}^{(LL)}=0,\quad(47)$$

$$|k_{s\mp n}{}^T|^2-|\boldsymbol{\kappa}_s|^2-|\boldsymbol{\kappa}_n|^2\pm2|\boldsymbol{\kappa}_s||\boldsymbol{\kappa}_n|\cos\psi_{T_{s\mp n}}{}^{(LL)}=0,\quad(48)$$

are obtained, respectively. From Eqs. (47) and (48), we conclude that light- and sound-type radiations are uniquely defined by the interaction angles $\psi_{T_{s\mp n}}{}^{(LL)}$ and $\psi_{L_{s\mp n}}{}^{(LL)}$. These angles are different for a given set of plasma parameters.

An important conclusion can be drawn from the expressions above. When the propagation vectors of the interacting primary waves are perpendicular to each other, Eq. (47) is never satisfied, since $|k_{s\mp n}{}^L|^2-|\boldsymbol{\kappa}_s|^2-(\boldsymbol{\kappa}_n|^2\neq0$ always. Therefore, *when propagation vectors of two longitudinal primary waves intersect each other at right angles, sound-type radiations will not be generated.* For the same condition, however, light-type radiations can be generated.

The above theorem is valid regardless of whether the primary waves are unpolarized or polarized. Furthermore, it is to be remembered that the relative positions of the scattering direction vector $\mathbf{n}_r$ with plane $(\boldsymbol{\kappa}_s,\boldsymbol{\kappa}_n)$ are necessary conditions to be considered in the generation of scattering the two types of radiations.

It is possible to see from the resonance scattering conditions that the resonance interaction of two longitudinal waves, propagating in the same direction, can lead to the appearance of scattered waves of sound and light types. Indeed, in this case, the resonance conditions $\chi_{s\mp n}{}^{L,T}=0$ yield

$$\omega_e^2\mp 2\omega_s\omega_n\left[1-\left(1-\frac{\omega_e^2}{\omega_s^2}\right)^{1/2}\left(1-\frac{\omega_e^2}{\omega_n^2}\right)^{1/2}\right]=0\,, \tag{49}$$

$$\omega_e^2-(\omega_s^2+\omega_n^2-\omega_e^2)\left(1-\frac{v^2}{c^2}\right)\mp 2\omega_s\omega_n\left[\frac{v^2}{c^2}-\left(1-\frac{\omega_e^2}{\omega_s^2}\right)^{1/2}\left(1-\frac{\omega_e^2}{\omega_n^2}\right)^{1/2}\right]=0\,, \tag{50}$$

respectively. Scattered waves will radiate with combination frequencies, and their amplitudes will increase during propagation.

In the case of waves propagating in the same direction with the same frequencies, Eq. (49) becomes meaningless, whereas Eq. (50) is satisfied. That is to say, *two longitudinal primary waves propagating in the same direction with the same frequencies, when interacting, do not generate sound-type, but generate light-type emissions.*

An effective cross section $d\sigma$ will be obtained by dividing the energy density of the scattered wave with the energy flux of the incoming waves. The energy flux for each incoming primary wave is

$$\frac{1}{8\pi}\frac{v}{\omega_e}\frac{\omega}{\omega_e}\nu_0^2(\mathbf{n}_r\cdot\hat{\kappa})(\omega^2-\omega_e^2)^{1/2}.$$

The differential cross sections for transverse and longitudinal scattered waves represent the average power of light and sound type scattered per-unit-solid angle:

$$\left(\frac{d\sigma}{d\Omega}\right)_{T_{s\mp n}}^{(LL)}=2\frac{V_0^2\omega_e^2(\omega_s^2-\omega_e^2)}{(8\pi)^3\rho^{(0)}c^6}\left(\frac{c}{v}\right)^3\frac{(1-\epsilon_s^2)^{-1}(1-\epsilon_n^2)^{-1}\epsilon_{s\mp n}}{\epsilon_s(\mathbf{n}_r\cdot\hat{\kappa}_s)+(\nu_{0n}/\nu_{0s})^2\epsilon_n(\mathbf{n}_r\cdot\hat{\kappa}_n)}\nu_{0n}^2|\mathbf{S}_{s\mp n}{}^{(LL)}|^2\sin^2\theta_{s\mp n}{}^{(LL)}, \tag{51}$$

$$\begin{aligned}\left(\frac{d\sigma}{d\Omega}\right)_{L_{s\mp n}}^{(LL)}=&2\frac{V_0^2\omega_e^2(\omega_s^2-\omega_e^2)}{(8\pi)^3\rho^{(0)}c^6}\frac{(1-\epsilon_s^2)^{-1}(1-\epsilon_n^2)^{-1}\epsilon_{s\mp n}}{\epsilon_s(\mathbf{n}_r\cdot\hat{\kappa}_s)+(\nu_{0n}/\nu_{0s})^2\epsilon_n(\mathbf{n}_r\cdot\hat{\kappa}_n)}\nu_{0n}^2|\mathbf{S}_{s\mp n}{}^{(LL)}|^2\cos^2\theta_{s\mp n}{}^{(LL)}\\&+\frac{V_0(\omega_s^2-\omega_e^2)r}{(8\pi)^2\rho^{(0)}v^4}\frac{(1-\epsilon_s^2)^{-1}(1-\epsilon_n^2)^{-1}\epsilon_{s\mp n}}{\epsilon_s(\mathbf{n}_r\cdot\hat{\kappa}_s)+(\nu_{0n}/\nu_{0s})^2\epsilon_n(\mathbf{n}_r\cdot\hat{\kappa}_n)}\nu_{0n}^2|\mathbf{S}_{s\mp n}{}^{(LL)}||T_{s\mp n}{}^{(LL)}|\cos\theta_{s\mp n}{}^{LL}\cos\phi_{s\mp n}{}^{LL}.\end{aligned} \tag{52}$$

The conversion cross section $(d\sigma/d\Omega)_{T_{s\mp n}}{}^{(LL)}$ is important in interpretations of the theory of radio outbursts from the sun. The generation of plasma waves in the isotropic chromosphere and corona is of interest in connection with sporadic solar radio emission, only when these longitudinal waves can be efficiently transformed into transverse (radio) waves. In a homogeneous plasma (where the first-order perturbation equations are valid), this transformation occurs only through scattering of longitudinal waves from the plasma medium. However, in an inhomogeneous plasma, the efficiency of the transformation is increased, due to interactions between plasma waves.

Case II. Transverse-Transverse Primary Waves

In the formulations, $\mathbf{T}_{s\mp n}$ disappears because of vanishing η_{1s} and η_{1n}. The radiation intensities have the following forms:

$$I_{T_{s\mp n}}{}^{(TT)}=\frac{V_0^2\omega_e^2(\omega_s^2-\omega_e^2)}{(8\pi)^4\rho^{(0)}c^5}(1-\epsilon_s^2)(1-\epsilon_n^2)\epsilon_{s\mp n}(\hat{\nu}_s\cdot\hat{\nu}_n)^2\nu_{0s}^2\nu_{0n}^2|\mathbf{S}_{s\mp n}{}^{(TT)}|^2\sin^2\theta_{s\mp n}{}^{(TT)}, \tag{53}$$

$$I_{L_{s\mp n}}{}^{(TT)}=\frac{V_0^2(\omega_s\mp\omega_n)^2(\omega_s^2-\omega_e^2)}{2(8\pi)^3\rho^{(0)}v^5}\left(\frac{v}{c}\right)^2(1-\epsilon_s^2)(1-\epsilon_n^2)\epsilon_{s\mp n}(\hat{\nu}_s\cdot\hat{\nu}_n)^2\nu_{0s}^2\nu_{0n}^2|\mathbf{S}_{s\mp n}{}^{(TT)}|^2\cos^2\theta_{s\mp n}{}^{(TT)}, \tag{54}$$

where

$$S_{s\mp n}{}^{(TT)}=\left[\left(\frac{\omega_n^2-\omega_e^2}{\omega_s^2-\omega_e^2}\right)^{1/2}\hat{\kappa}_n\mp(1-\delta_s{}^n)\hat{\kappa}_s\right]. \tag{55}$$

These expressions assume that scattering resonance conditions are satisfied. Similar expressions to Eqs. (47) and (48) can be written to define angles of interactions $\psi_{L_{s\mp n}}{}^{(TT)}$ and $\psi_{T_{s\mp n}}{}^{(TT)}$ for a given set of plasma parameters.

If the primary waves are plane polarized, and if they interact with polarization planes at right angles to each other ($\hat{\nu}_s\cdot\hat{\nu}_n=0$), *there will be no scattering radiation, even though resonance conditions are realized.* This is the celebrated optical theorem of Fresnel-Arago, generalized for a plasma field. Indeed, Fresnel and Arago investigated the interference of polarized rays of light and found that two rays polarized at right angles to each other never interfere (from which they concluded that light vibrations must be transverse).[14]

Now, assuming that linearly polarized primary waves are not at right angles to each other and are traveling in the same direction, the resonance conditions can be satisfied, and light- and sound-type emission are obtained by satisfying

$$\omega_e^2\mp 2\omega_s\omega_n\left[1-\left(1-\frac{\omega_e^2}{\omega_s^2}\right)^{1/2}\left(1-\frac{\omega_e^2}{\omega_n^2}\right)^{1/2}\right]=0\,, \tag{56}$$

$$\left(\frac{v}{c}\right)^2\omega_e^2+(\omega_s^2+\omega_n^2-\omega_e^2)\left[1-\frac{v^2}{c^2}\left(\mp 2\omega_s\omega_n\right)1-\frac{v^2}{c^2}\left(1-\frac{\omega_e^2}{\omega_s^2}\right)^{1/2}\left(1-\frac{\omega_e^2}{\omega_n^2}\right)^{1/2}\right]=0\,, \tag{57}$$

respectively. *It is furthermore to be noted, that when the two primary waves are propagating with the same frequencies* ($\omega_s=\omega_n$), *there will be no light-type emissions*, since Eq. (56) becomes meaningless. However, for the same condition, a sound-type emission is possible.

When the primary waves are interacting with propagation vectors at right angles to each other, there can be no light-type emission.

The energy flux carried by an incoming primary wave is $(c/16\pi)\nu_0^2(\mathbf{n}_r\cdot\hat{\kappa})(\omega^2-\omega_e^2)^{1/2}$. The differential cross sections are written in a similar fashion to the previous case:

$$\left(\frac{d\sigma}{d\Omega}\right)_{T_{s\mp n}}^{(TT)}=2\frac{V_0^2\omega_e^2(\omega_s^2-\omega_e^2)}{(8\pi)^3\rho^{(0)}c^6}\,\frac{(1-\epsilon_s^2)(1-\epsilon_n^2)\epsilon_{s\mp n}(\hat{\nu}_s\cdot\hat{\nu}_n)^2}{\epsilon_s(\mathbf{n}_r\cdot\hat{\kappa}_s)+\epsilon_n(\nu_{0n}/\nu_{0s})^2(\mathbf{n}_r\cdot\hat{\kappa}_n)}\nu_{0n}^2|\mathbf{S}_{s\mp n}{}^{(TT)}|^2\sin^2\theta_{s\mp n}{}^{(TT)}, \tag{58}$$

$$\left(\frac{d\sigma}{d\Omega}\right)_{L_{s\mp n}}^{(TT)}=\frac{V_0^2(\omega_s\mp\omega_n)^2(\omega_s^2-\omega_e^2)}{(8\pi)^2\rho^{(0)}v^6}\left(\frac{v}{c}\right)^3\frac{(1-\epsilon_s^2)(1-\epsilon_n^2)\epsilon_{s\mp n}(\hat{\nu}_s\cdot\hat{\nu}_n)^2}{\epsilon_s(\mathbf{n}_r\cdot\hat{\kappa}_s)+\epsilon_n(\nu_{0n}/\nu_{0s})^2(\mathbf{n}_r\cdot\hat{\kappa}_n)}\nu_{0n}^2|\mathbf{S}_{s\mp n}{}^{(TT)}|^2\cos^2\theta_{s\mp n}{}^{(TT)}. \tag{59}$$

Special cases, such as interacting electromagnetic and transverse plasma waves, or two electromagnetic waves in a plasma medium, can be obtained from the above general expressions.

Case III. Longitudinal-Transverse Primary Waves

Assuming that s and n indices associate with longitudinal and transverse primary waves, respectively, the intensity expressions for radiating fields become:

$$I_{T_{s\mp n}}{}^{(LT)}=\frac{V_0^2\omega_e^2(\omega_s^2-\omega_e^2)}{(8\pi)^4\rho^{(0)}c^5}\left(\frac{v}{c}\right)^{-2}\epsilon_{s\mp n}\nu_{0s}^2\nu_{0n}^2|\mathbf{S}_{s\mp n}{}^{(LT)}|^2\sin^2\theta_{s\mp n}{}^{(LT)}, \tag{60}$$

$$I_{L_{s\mp n}}{}^{(LT)}=\frac{V_0^2(\omega_s\mp\omega_n)^2(\omega_s^2-\omega_e^2)}{2(8\pi)^3\rho^{(0)}v^5}\epsilon_{s\mp n}\nu_{0s}^2\nu_{0n}^2|\mathbf{S}_{s\mp n}{}^{(LT)}|^2\cos^2\theta_{s\mp n}{}^{(LT)}+\frac{V_0(\omega_s^2-\omega_e^2)r}{2(8\pi)^3\rho^{(0)}v^3}\epsilon_{s\mp n}\nu_{0s}^2\nu_{0n}^2|\mathbf{S}_{s\mp n}{}^{(LT)}|^2\cos\theta_{s\mp n}{}^{(LT)}\cos\phi_{s\mp n}{}^{(LT)}, \tag{61}$$

where

$$\mathbf{S}_{s\mp n}{}^{(LT)}=\left\{2\hat{\nu}_n+\frac{\omega_s}{\omega_n}(\hat{\nu}_s\cdot\hat{\nu}_n)\left[\left(\frac{\omega_n^2-\omega_e^2}{\omega_s^2-\omega_e^2}\right)^{1\,2}\left(\frac{v}{c}\right)\hat{\kappa}_n\mp(1-\delta_s{}^n)\hat{\kappa}_s\right]\right\}, \tag{62}$$

$$\mathbf{T}_{s\mp n}{}^{(LT)}=\left\{2\hat{\nu}_n-\left[\mp(1-\delta_s{}^n)\left(\frac{\omega_s}{\omega_n}\right)(\hat{\nu}_s\cdot\hat{\nu}_n)\right]\hat{\kappa}_s\right\}. \tag{63}$$

Again, one can substitute in Eqs. (47) and (48) the propagation vectors of the primary waves of this plane polarized to their relative positions. Light- and sound-type radiation will be generated, provided that other radiation condi-

[14] A. Fresnel, Ann. Chim. Phys. 2, 1 (1816), 2396 Oeuvers, Vol. 1, 39–129.

tions are realized. The primary waves propagate in the same direction; light- and sound-type radiations are possible if

$$\omega_e^2-\left(1-\frac{v^2}{c^2}\right)\omega_s^2\mp 2\omega_s\omega_n\left[\frac{v}{c}-\left(\frac{\omega_e^2}{\omega_s^2}\right)^{1/2}\left(1-\frac{\omega_e^2}{\omega_n^2}\right)^{1/2}\right]=0, \tag{64}$$

$$\left(\frac{v}{c}\right)^2\omega_e^2+\left(1-\frac{v^2}{c^2}\right)\omega_n^2\mp 2\omega_s\omega_n\left[1-\frac{v}{c}\left(1-\frac{\omega_e^2}{\omega_s^2}\right)^{1/2}\left(1-\frac{\omega_e^2}{\omega_n^2}\right)^{1/2}\right]=0, \tag{65}$$

are satisfied, respectively. In contrast to the previous case, primary waves propagating in the same direction with equal frequencies will interact and generate the two types of emissions.

Repeating the same procedures as in the previous cases, differential cross sections for the longitudinal-transverse primary waves are written as follows:

$$\left(\frac{d\sigma}{d\Omega}\right)_{T_{n\mp n}}^{(LT)}=2\frac{V_0^2\omega_e^2(\omega_s^2-\omega_e^2)}{(8\pi)^3\rho^{(0)}c^6}\left(\frac{v}{c}\right)^{-2}\frac{\epsilon_{s\mp n}}{(v/c)\epsilon_s(\mathbf{n}_r\cdot\hat{\kappa}_s)+\epsilon_n(\nu_{0n}/\nu_{0s})^2(\mathbf{n}_r\cdot\hat{\kappa}_n)}\nu_{0n}^2|\mathbf{S}_{s\mp n}^{(LT)}|^2\sin^2\theta_{s\mp n}^{(LT)}, \tag{66}$$

$$\left(\frac{d\sigma}{d\Omega}\right)_{L_{s\mp n}}^{(LT)}=\frac{V_0^2(\omega_s\mp\omega_n)^2(\omega_s^2-\omega_e^2)}{(8\pi)^2\rho^{(0)}v^6}\left(\frac{v}{c}\right)\frac{\epsilon_{s\mp n}}{(v/c)\epsilon_s(\mathbf{n}_r\cdot\hat{\kappa}_s)+\epsilon_n(\nu_{0n}/\nu_{0s})^2(\mathbf{n}_r\cdot\hat{\kappa}_n)}\nu_{0n}^2|\mathbf{S}_{s\mp n}^{(LT)}|^2\cos^2\theta_{s\mp n}^{(LT)}$$
$$+\frac{v_0(\omega_s^2-\omega_e^2)r}{(8\pi)^2\rho^{(0)}v^4}\left(\frac{v}{c}\right)\frac{\epsilon_{s\mp n}}{(v/c)\epsilon_s(\mathbf{n}_r\cdot\hat{\kappa}_s)+\epsilon_n(\nu_{0n}/\nu_{0s})^2(\mathbf{n}_r\cdot\hat{\kappa}_n)}$$
$$\times\nu_{0n}^2|\mathbf{S}_{s\mp n}^{(LT)}||\mathbf{T}_{s\mp n}^{(LT)}|\cos\theta_{s\mp n}^{(LT)}\cos\phi_{s\mp n}^{(LT)}, \tag{67}$$

V. SUMMARY AND DISCUSSIONS

We have investigated light- and sound-type emissions from nonlinear fluctuations of an electron plasma, and related some of the results with the well-known results of classical electromagnetic and hydrodynamic theories in the limiting cases. The analysis is performed for second-order perturbations which contain radiations of quadrupole type as well as radiations of lower orders. It should be kept in mind that the second-approximation effects are small when compared with those of the first approximation.

Mean energies emitted per unit time from the plasma field can be looked upon as radiating energies of light and sound waves. Therefore, intensity expressions refer only to the energies which actually escape as sound and light, and to their directional distributions. Due to the three different possible interactions, six types of intensity expressions are in order; three of these radiations are sound, and the other three are light-type radiations. We have briefly referred to transverse and longitudinal components of radiation intensities as light and sound.

The intensity expressions of scattered light and sound are proportional to the same form

$$\frac{(\text{volume})^2\ (\text{frequency})^4\ (\text{source term})^2}{(\text{equilibrium density})^1\ (\text{characteristic velocity})^5}, \tag{68}$$

where frequency, source, and velocity parameters depend on the type of interacting primary waves. In the sound radiation field, a dipole-like radiation term enters into the expressions. In the general expression of (68), amplitude terms $\nu_{0s}^2\nu_{0n}^2$ are absorbed into the source term to make it dimensionally equivalent to that of a stress term.

In these six types of radiation expressions, the velocity for pure sound radiation is the adiabatic sound velocity v; for a pure light radiation, it is the velocity of light c. Intermediate types of radiation will contain (v/c), $(v/c)^2$, $\cdots$, etc., as multipliers of the expression of the pure light radiation. Here, we use the term "pure" for cases where only *one* type of velocity enters into the expressions.

Similar studies have been done in the field of acoustics, of quadrupole sound radiation. Case III is particularly interesting in this sense, since it yields the above-mentioned acoustic solutions as a special case. It is evident from Eq. (1) to (5), that when the electronic charge vanishes, the hydrodynamic field becomes uncoupled from Maxwell's field. If the electronic charge now goes to zero in the intensity expression of the scattered longitudinal wave, one obtains:

$$I_{L_{s\mp n}}^{(LT)}=\frac{V_0^2\omega_s^2(\omega_s\mp\omega_n)^2}{(8\pi)^4\rho^{(0)}v^5}|\nu_{0s}\nu_{0n}\mathbf{S}_{s\mp n}^{(LT)}|^2\sin^2\theta_{s\mp n}^{(LT)}$$
$$+\frac{V_0\omega_s^2r}{2(8\pi)^3\rho^{(0)}v^3}|\nu_{0s}\nu_{0n}\mathbf{S}_{s\mp n}^{(LT)}|^2$$
$$\times\cos\theta_{s\mp n}^{(LT)}\cos\phi_{s\mp n}^{(LT)}, \tag{69}$$

which can be compared with known acoustic results.[15]

[15] M. J. Lighthill, Proc. Roy. Soc. (London) **A211**, 564 (1952).

Indeed, the first term on the right-hand side of the last equation is comparable to acoustic quadrupole radiation, whereas the second term represents a dipole-like radiation. The main difference between this result and that of the earlier acoustic work is the appearance here of combination frequencies.

We have formulated all possible radiations resulting from the three types of interactions. Some useful conclusions can be drawn from these results:

(1) It is seen that in some cases, depending on the nature of interacting primary waves, sound and light emissions become prohibitive if they interact at right angles to each other (the interaction angle is to be understood as the angle between the propagation vectors of the primary waves).

(a) When (both) primary waves are longitudinal, no sound-type radiation is possible but a light-type emission will be possible. This theorem is general and is true for polarized, unpolarized primary waves.

(b) If the two transverse primary plasma waves are interacting at right angles to each other, the light-type emission is prohibited but sound emission is possible. Obviously this excludes the case wherein the plane polarized primary waves interact with the polarization planes at right angles to each other.

(c) When primary waves are longitudinal and transverse both sound- and light-type emissions are possible.

Furthermore, one can say that the above statements are true for primary waves having different or equal frequencies.

(2) From general expressions of resonance scattering equations (47), (48), we are able to determine radiation conditions for the primary waves traveling on the same direction.

(a) When both primary waves are longitudinal, sound- and light-type emissions are possible. If they have the same frequencies, sound emission is impossible.

(b) If primary waves are transverse waves, both sound- and light-type emissions are possible. When they have the same frequencies there is no light-type emission, and a sound-type emission is possible (again for the interaction of plane polarized primary waves with polarization planes at right angles, the two types of radiations are impossible).

(c) If primary waves are longitudinal and transverse, both sound- and light-type emissions are possible for different and equal frequencies.

(3) Generally, if primary waves are interacting with an arbitrary angle (other than 0° and 90°) both sound- and light-type emissions are possible for different and equal frequencies.

(4) Limiting cases to the general results, such as interacting sound waves, interacting sound and electromagnetic waves, or interacting electromagnetic-electromagnetic waves in a plasma medium can be obtained simply by converting primary plasma waves into ordinary electromagnetic and sound waves. This can be done easily by equating Langmuir electron frequency ω_e to zero in the primary wave numbers κ_s, κ_n.

(5) We find in the second-order perturbations that oscillations with frequencies $\omega_s \mp \omega_n$ enter into the radiation expressions. $\omega_s \mp \omega_n$ type combination frequencies include self-interaction frequencies (double frequencies) $2\omega_1$, $2\omega_2$ as well as interaction frequencies $\omega_1-\omega_2$, $\omega_1+\omega_2$. In higher approximations (for instance $n=3$) combination frequencies will appear as the sums and differences of more than two initial frequencies $3\omega_1$, $3\omega_2$, $2\omega_1+\omega_2$, $\omega_1-2\omega_2$, $\omega_2-2\omega_1$, $\omega_1+2\omega_2$. However, the combination frequencies will include some terms which coincide with the original frequencies $\omega_1=\omega_1+\omega_2-\omega_2$, $\omega_2=\omega_2+\omega_1-\omega_1$. Carrying this to higher approximations, combination frequencies for an nth order will be in the form of

$$\omega^{(n)} = k\omega_1 \mp l\omega_2\,, \tag{70}$$

where k and l are integral numbers, $n=k+l$. Another important aspect of the problem can be developed by investigating the interaction of an arbitrary number of primary waves. That is to say, Eq. (25) must be reconsidered in the form

$$\mathbf{E}^{(1)} = \sum_{j=1}^{n} \mathbf{v}_j \cos(\omega_j t - \boldsymbol{\kappa}_j \cdot \mathbf{r})\,, \tag{71}$$

which includes n arbitrary interacting primary waves.